Fermi National Accelerator Laboratory

FERMILAB-FN-611

Beam Loading and Potential Well Distortion for a Partially Filled Ring

S.A. Bogacz

Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

November 1993

Operated by Universities Research Association Inc. under Contract No. DE-AC02-76CHO3000 with the United States Department of Energy

BEAM LOADING AND POTENTIAL WELL DISTORTION FOR A PARTIALLY FILLED RING

S.A. Bogacz

Accelerator Physics Department,
Fermi National Accelerator Laboratory*
P.O. Box 500, Batavia, IL 60510

October 1993

A novel analytic treatment of the multi-bunch motion for a partially filled ring is presented. We start with a standard set of wake field-coupled equations of motion describing a train of M consecutive bunches in a storage ring of harmonic number N, where the wake field effects are separated into: the beam loading force, the incoherent tune shift due to potential well distortion and the coherent multi-bunch coupling. Unlike in the case of completely filled ring, now, the first two quantities vary from bunch to bunch. Here, we evaluate both quantities analytically (using contour integration technique) for a general situation of a partially filled ring (M < N), where individual bunches are mutually interacting via wake fields generated by resonant structures. Resulting simple analytic formulas describe the beam loading force and the incoherent tune shift experienced by a given bunch within the train, as a function of the resonant frequency, ω_r, and the quality factor of the coupling impedance, Q. The first formula reveals resonant frequency regions in the vicinity of the integer multiples of the r.f. frequency, $N\omega_o$, where the beam loading response is still equal for all bunches (its absolute value scales as M). It also identifies the second (denser) set of characteristic resonant frequencies, spaced by the multiples of $N\omega_o/M$, at which the beam loading force is not only bunch independent, but also considerably smaller (it scales as MQ^{-2}). Conversely, our analytic formula identifies frequency regions, where bunch-to-bunch variation of the beam loading force is the strongest (ω_r at odd multiples of $N\omega_o/2M$). Similarly, an analogous analytic formula describing the amount of incoherent synchrotron tune shift suffered by different bunches was derived. Both analytic expressions give one an insight into various optimizing schemes; e.g. to modify the existing configuration of parasitic cavity resonances (via frequency tuning), so that the resulting bunch-to-bunch spread of the beam loading force and/or the synchrotron tune spread could be instrumental in stabilizing (via Landau damping) some unstable modes of the coupled bunch instability. A number of other possible applications of the presented formalism emerge from the fact that for a given configuration of cavity resonances one can get immediately a simple quantitative answer in terms of the beam loading and the synchrotron tune shift experienced by each bunch along the train.

---

*Operated by the Universities Research Association, Inc., under a contract with the U.S. Department of Energy

1. INTRODUCTION

While coupled multi-bunch motion for a symmetric configuration of populated buckets in a storage ring has been extensively studied and the stability problem has a closed analytic solution[1], a fully populated ring is rarely the case for any operational mode of a realistic synchrotron. As a beam is injected and extracted to and from a storage ring there is usually a gap of missing bunches to accommodate injection/extraction kickers. Even a small gap breaks down the symmetry of a coupled multi-bunch motion. Quite often, an extremely non-symmetric situation is present in high energy colliders, where a relatively short train of bunches is being accelerated in a long storage ring; e.g. during bunch coalescing.

Here, we present a rigorous treatment of the beam loading and potential well distortion effects for a general non-symmetric configuration of populated buckets. We formulate the problem in the framework of a system of differential equations of motion for individual bunches coupled via wake fields. First, we extract non-symmetric beam loading and potential well distortion terms from the rest of the multi-bunch coupling. The core of this paper is an analytic method, involving contour integration in complex frequency domain, which yields a pair of closed expressions describing the beam loading and the potential well distortion effects, driven by a general resonant impedance, for the case of partially filled ring.

Both quantities, the beam loading force and the incoherent synchrotron tune shift, are expressed in terms of explicit functions of: the bunch index, the resonant frequency and the quality factor of the impedance peak. Superimposing many parasitic cavity modes one can use the above formulas to choose appropriate tuning of existing configuration of parasitic modes to increase stability of the coupled bunch motion.

2. COUPLED MULTI-BUNCH MOTION

We assume a storage ring of a harmonic number N populated by a train of M consecutive bunches, where $M \leq N$. We will confine our consideration to the dipole mode of the longitudinal motion only. Therefore, it will be sufficient to model each bunch as a macro particle combining intensity of the entire bunch. To describe a coupled motion of a system of M bunches one can represent a state of the system at a given time by the following vector in the M-dimensional configuration space $\mathbf{R}^M$

$$|y(t)\rangle = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ y_M(t) \end{pmatrix} , \tag{2.1}$$

where the n-th component of the above vector describes the longitudinal coordinate (e.g. longitudinal displacement with respect to the center of a bucket) of the n-th bunch.

Collective synchrotron motion of the system on M bunches coupled via wake fields can be described by the following set of equations of motion[1,2]

$$\frac{\partial^2}{\partial t^2} y_n(t) + \omega^2 y_n(t) = A \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W'\left(-(k + \tfrac{m-n}{N})T_0 + \tfrac{1}{c}\left[y_n(t) - y_m\left(t - (k + \tfrac{m-n}{N})T_0\right)\right]\right) ,$$

where (2.2)

$$A = \frac{N_0 r_0 \eta \omega_0}{2\pi\gamma} .$$

Here W' is the time derivative of the wake function, ω is the unperturbed synchrotron frequency, η is the revolution frequency dispersion function, c is the velocity of light, r_0 is the classical proton radius, ω_0 is the revolution frequency and T_0 is the revolution period.

The argument of the wake function in the right hand side of Eq.(2.2) can be separated into a large part given by the bunch separation and a small correction of the order of the relative bunch displacement due to the synchrotron motion. Taylor expansion of the wake function with respect to the difference of bunch displacements (up to the linear term) allows one to rewrite the set of equations of motion as follows

$$\frac{\partial^2}{\partial t^2} y_n(t) + \omega^2 y_n(t) = A \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W'\Big(-(k + \tfrac{m-n}{N})T_0\Big) + \tag{2.3}$$

$$+ A\,\frac{1}{c} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W''\Big(-(k + \tfrac{m-n}{N})T_0\Big) \Big[y_n(t) - y_m\Big(t - (k + \tfrac{m-n}{N})T_0\Big)\Big] \ .$$

We can identify the first term in the right hand side of Eq.(2.3) with the beam loading force, which drives the n-th bunch. It explicitly depends on the bunch index n. We will denote it by f_n.

$$f_n = A \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W'\Big(-(k + \tfrac{m-n}{N})T_0\Big) \ . \tag{2.4}$$

Furthermore, a term proportional to $y_n(t)$ in the right hand side of Eq.(2.3) can be absorbed by the synchrotron frequency. This is known as the incoherent synchrotron frequency shift due to the potential well distortion; it will be denoted by $\Delta\omega_n^2$. This term can be absorbed to redefine the perturbed synchrotron frequency corrected according to the following expression

$$\omega_n^2 = \omega^2 - \Delta\omega_n^2$$

where (2.5)

$$\Delta\omega_n^2 = A\,\frac{1}{c} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W''\Big(-(k + \tfrac{m-n}{N})T_0\Big)$$

Here, ω_n – the corrected synchrotron frequency of the n-th bunch explicitly depends on the bunch index n. Both f_n and $\Delta\omega_n^2$ will be evaluated explicitly in the next section.

Now the set of equations of motion, Eq.(2.2), can be rewritten as follows

$$\frac{\partial^2}{\partial t^2} y_n(t) + \left(\omega^2 - \Delta\omega_n^2\right) y_n(t) = f_n + \tag{2.6}$$

$$- A \frac{1}{c} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} W''\left(-(k + \frac{m-n}{N})T_0\right) y_m\left(t - (k + \frac{m-n}{N})T_0\right),$$

where the last term in the right hand side of Eq.(2.6) represents pure coherent multi-bunch wake field coupling term, which may result in a collective synchrotron motion of bunches - a multi-bunch instability.

The resulting set of equations of motion, Eq.(2.6), along with a convenient representation of the wake field coupling in terms of the longitudinal impedance, will be analyzed in the complex frequency domain later in the paper.

3. BEAM LOADING AND INCOHERENT TUNE SHIFT – FULL VS PARTIALLY FILLED RING

To evaluate both the beam loading term and the incoherent synchrotron frequency shift due to the potential well distortion it is convenient to express them in terms of the longitudinal coupling impedance. The time derivative of the wake function is related to the longitudinal impedance via the inverse Fourier transform as follows

$$W'(t) = \frac{c}{2\pi} \int_{-\infty}^{\infty} d\omega \; e^{i\omega t} \, Z_{\|}(\omega) \quad . \tag{3.1}$$

Similar relationship holds for the second time derivative of the wake function

$$W''(t) = -\frac{c}{2\pi i} \int_{-\infty}^{\infty} d\omega \; \omega \; e^{i\omega t} \, Z_{\|}(\omega) \quad . \tag{3.2}$$

One may substitute Eq.(3.1) and (3.2) into Eq.(2.4) and (2.5) respectively. The resulting expressions are written as follows

$$f_n = \frac{Ac}{2\pi} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} \int_{-\infty}^{\infty} d\omega \; e^{-i\omega(k + \frac{m-n}{N})T_0} \, Z_{\|}(\omega) \ , \tag{3.3}$$

and

$$\Delta\omega_n^2 = \frac{A}{2\pi i} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{M-1} \int_{-\infty}^{\infty} d\omega \; \omega e^{-i\omega(k + \frac{m-n}{N})T_0} \, Z_{\|}(\omega) \ . \tag{3.4}$$

Infinite summation over k can be carried out explicitly using a trivial version of the Poisson sum identity:

$$\sum_{k=-\infty}^{\infty} e^{-i\omega k T_0} = \omega_0 \sum_{p=-\infty}^{\infty} \delta(\omega - p\omega_0) \tag{3.5}$$

Substituting Eq.(3.5) in Eqs.(3.3) and (3.2) one can also carry out integration over ω. The resulting expressions are given by

$$f_n = \omega_0 \frac{Ac}{2\pi} \sum_{p=-\infty}^{\infty} \sum_{m=0}^{M-1} Z_{\parallel}(p\omega_0)\ e^{2\pi i p \frac{n-m}{N}} , \tag{3.6}$$

and

$$\Delta\omega_n^2 = \omega_0 \frac{A}{2\pi i} \sum_{p=-\infty}^{\infty} \sum_{m=0}^{M-1} (p\omega_0)\ Z_{\parallel}(p\omega_0)\ e^{2\pi i p \frac{n-m}{N}} . \tag{3.7}$$

Applying a simple sum identity, Eq.(A1), to Eqs.(3.6) and (3.7) one can rewrite them in the following form

$$f_n = \omega_0 \frac{Ac}{2\pi} \sum_{q=-\infty}^{\infty} \sum_{\ell=0}^{N-1} e^{2\pi i \ell \frac{n}{N}}\ Z_{\parallel}\Big((Nq+\ell)\omega_0\Big) \sum_{m=0}^{M-1} e^{-2\pi i \ell \frac{m}{N}}, \tag{3.8}$$

and

$$\Delta\omega_n^2 = \omega_0 \frac{A}{2\pi i} \sum_{q=-\infty}^{\infty} \sum_{\ell=0}^{N-1} e^{2\pi i \ell \frac{n}{N}} \Big[(Nq+\ell)\omega_0\Big]\ Z_{\parallel}\Big((Nq+\ell)\omega_0\Big) \sum_{m=0}^{M-1} e^{-2\pi i \ell \frac{m}{N}}, \tag{3.9}$$

The last summation (over m) in Eqs.(3.8) and (3.9) can be carried out explicitly, Eq.(A3) and (A5), (see Appendix A). The resulting formula is written as follows

$$B_\ell = \sum_{m=0}^{M-1} e^{-2\pi i \ell F(m,N)} = e^{\pi i \frac{\ell}{N}(M-1)} \frac{\sin \pi \frac{\ell M}{N}}{\sin \pi \frac{\ell}{N}} , \qquad \ell = 0 \ldots N-1 . \tag{3.10}$$

One can immediately see that in case of the full ring (M = N) the above expression, Eq.(3.10), reduces to the following simple form

$$B_\ell = N\,\delta_{\ell,0} , \qquad \ell = 0 \ldots N-1 . \tag{3.11}$$

Substitution Eq.(3.11) into Eqs.(3.8) and (3.9) and carrying summation over ℓ yields the following set of expressions for the full ring

$$(f)_{\mathrm{full}} = \omega_0 \frac{Ac}{2\pi} N \sum_{q=-\infty}^{\infty} Z_{\|}(Nq\omega_0) \ , \tag{3.12}$$

and

$$(\Delta\omega^2)_{\mathrm{full}} = \omega_0 \frac{A}{2\pi i} N \sum_{q=-\infty}^{\infty} (Nq\omega_0)\, Z_{\|}(Nq\omega_0) \ . \tag{3.13}$$

We notice in passing that in the above expressions, Eqs.(3.12) and (3.13), the bunch index, n, is no longer present. This simplicity, granted by the symmetry of the bunch configuration is inherent to the full ring case only. For the general partially filled ring case ($M < N$) both quantities: f_n and $\Delta\omega_n^2$, given by Eqs.(3.8) and (3.9) vary with the bunch location within the sequence. In the next two sections we will derive analytically a simple set of closed formulas describing f_n and $\Delta\omega_n^2$ in the case of partially filled ring ($M < N$) driven by a general resonant impedance.

4. BEAM LOADING EFFECTS

We wish to evaluate the general beam loading function, f_n, given by Eq.(3.6), where $n = 0,..., M - 1$ for the case of partially filled ring ($M < N$). Summing explicitly over m, one can rewrite Eq.(3.6) into the following convenient form

$$f_n = \omega_0 \frac{Ac}{2\pi} \sum_{p=-\infty}^{\infty} e^{2\pi i (p\omega_0)\frac{n}{N\omega_0}} \; Z_{\|}(p\omega_0) \; \frac{1 - e^{-2\pi i (p\omega_0)\frac{M}{N\omega_0}}}{1 - e^{-2\pi i (p\omega_0)\frac{1}{N\omega_0}}} = \sum_{p=-\infty}^{\infty} F(p\omega_0) \tag{4.1}$$

The last part of Eq.(4.1) highlights generic sampling structure of the above expression. Applying the following form of the Poisson sum identity

$$\sum_{p=-\infty}^{\infty} F(p\omega_0) = \sum_{q=-\infty}^{\infty} \frac{1}{\omega_0} \int_{-\infty}^{\infty} d\omega \; e^{2\pi i q \frac{\omega}{\omega_0}} \; F(\omega) \, , \tag{4.2}$$

to Eq.(4.1) one can express it in the following form

$$f_n = iAc \sum_{q=-\infty}^{\infty} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \; Z_{\|}(\omega) \; \frac{e^{2\pi i (q + \frac{n}{N}) \frac{\omega}{\omega_0}} - e^{2\pi i (q - \frac{M-n}{N}) \frac{\omega}{\omega_0}}}{1 - e^{-2\pi i \frac{\omega}{N\omega_0}}} \; . \tag{4.3}$$

Introducing two kinds of generic integrals, namely:

$$I^{+}(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \; Z_{\|}(\omega) \; \frac{e^{2\pi i k \frac{\omega}{N\omega_0}}}{1 - e^{-2\pi i \frac{\omega}{N\omega_0}}} \; , \qquad k \geq 0 \tag{4.4}$$

and

$$I^-(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, Z_{\|}(\omega)\, \frac{e^{-2\pi i k \frac{\omega}{N\omega_0}}}{1 - e^{-2\pi i \frac{\omega}{N\omega_0}}} \quad , \qquad k \geq 0 \tag{4.5}$$

one can express the beam loading term, Eq.(4.3), in the following compact form

$$f_n = iAc \left\{ I^+(n) - I^-(k) + \sum_{q=1}^{\infty} \left[I^+(Nq + n) + I^-(Nq - n) \right.\right. +$$

$$\left.\left. - I^+(Nq - (M - n)) - I^-(Nq + (M - n)) \right] \right\} \quad , \tag{4.6}$$

Assuming general form of the longitudinal impedance of a resonant structure, given by the following standard expression:

$$Z_{\|}(\omega) = \frac{R}{1 + iQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} \quad , \qquad Q >> 1 \tag{4.7}$$

where R is the shunt impedance, Q is the quality factor of the resonator and ω_r is its resonant frequency, one can evaluate integrals $I^+(k)$ and $I^-(k)$ explicitly via contour integration (see Appendix B). The resulting expressions given by Eqs.(B5) and (B6) are summarized as below

$$I^+(k) = \frac{1}{2}\,\omega_0\, \frac{1}{2\pi i}\, N \sum_{p=-\infty}^{\infty} Z_{\|}(Np\omega_0) - i\,\frac{R}{2Q}\left[\frac{\omega_+\, e^{2\pi i k \frac{\omega_+}{N\omega_0}}}{1 - e^{-2\pi i \frac{\omega_+}{N\omega_0}}} - \frac{\omega_-\, e^{2\pi i k \frac{\omega_-}{N\omega_0}}}{1 - e^{-2\pi i \frac{\omega_-}{N\omega_0}}} \right] \quad , \tag{4.8}$$

and

$$I^-(k) = -\frac{1}{2}\,\omega_0\, \frac{1}{2\pi i}\, N \sum_{p=-\infty}^{\infty} Z_{\|}(Np\omega_0) \quad , \tag{4.9}$$

where the singularities of $Z_{\|}(\omega)$ are defined by the following pair of complex poles $\omega_\pm$ located in the upper half plane

$$\omega_{\pm} = \omega_r(\pm 1 + i\delta), \qquad \delta = \frac{1}{2Q} << 1 \quad . \tag{4.10}$$

Substituting the above integrals, Eqs.(4.9)–(4.10) into Eq.(4.6) one can carry out remaining infinite summation over q, which converges to the following simple expression

$$\sum_{q=1}^{\infty} e^{2\pi i q \frac{\omega_{\pm}}{\omega_o}} = \frac{e^{2\pi i k \frac{\omega_{\pm}}{\omega_o}}}{1 - e^{2\pi i \frac{\omega_{\pm}}{\omega_o}}} \quad , \quad \text{since Im } \omega_{\pm} > 0. \tag{4.11}$$

Finally, one can rewrite Eq.(4.6) in the following form

$$f_n = \omega_0 \frac{Ac}{2\pi} N \sum_{p=-\infty}^{\infty} Z_{||}(Np\omega_0) + \tag{4.12}$$

$$+ Ac\frac{R}{2Q}\left[\frac{\omega_+ \, e^{2\pi i n \frac{\omega_+}{N\omega_o}}\left(1 - e^{2\pi i(N-M)\frac{\omega_+}{N\omega_o}}\right)}{\left(1 - e^{2\pi i \frac{\omega_+}{N\omega_o}}\right)\left(1 - e^{2\pi i \frac{\omega_+}{\omega_o}}\right)} - \frac{\omega_- \, e^{2\pi i n \frac{\omega_-}{N\omega_o}}\left(1 - e^{2\pi i(N-M)\frac{\omega_-}{N\omega_o}}\right)}{\left(1 - e^{2\pi i \frac{\omega_-}{N\omega_o}}\right)\left(1 - e^{2\pi i \frac{\omega_-}{\omega_o}}\right)}\right] .$$

The first term in Eq.(4.12) can be immediately identified with the beam loading term for a fully populated ring, $(f)_{full}$, given in the previous section by Eq.(3.12). Furthermore, $(f)_{full}$ was evaluated explicitly for our model resonant impedance, Eq.(4.7), at the end of the Appendix B (see Eq.(B.14)). The result can be summarized as follows

$$(f)_{full} = \frac{AcR\omega_o}{\pi}\left(\frac{\pi\omega_r}{N\omega_o}\right) N\,\delta^2 \, \frac{\left(\frac{\pi\omega_r}{N\omega_o}\right) - \frac{1}{2}\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2 \delta^2} \; . \tag{4.13}$$

Introducing a new function $\Gamma(\omega)$ defined by:

$$\Gamma(\omega) = \frac{e^{2\pi i n \frac{\omega}{N\omega_o}}\left(1 - e^{2\pi i(N-M)\frac{\omega}{N\omega_o}}\right)}{\left(1 - e^{2\pi i \frac{\omega}{N\omega_o}}\right)\left(1 - e^{2\pi i \frac{\omega}{\omega_o}}\right)}, \tag{4.14}$$

one can express f_n in the following compact form

$$f_n = (f)_{full} + Ac\frac{R}{2Q}\left[\omega_+ \Gamma(\omega_+) - \omega_- \Gamma(\omega_-)\right] \quad . \tag{4.15}$$

Employing symmetry of $\Gamma(\omega)$, namely

$$\Gamma(-\omega^*) = \Gamma^*(\omega), \tag{4.16}$$

and the fact that $\omega_- = -\omega_+^*$, one can rewrite Eq.(4.15) as follows

$$f_n = (f)_{full} + Ac\frac{R}{2Q}\, 2\mathrm{Re}\left[\omega_+ \Gamma(\omega_+)\right] \quad . \tag{4.17}$$

Cutting through some tedious algebra explicit expressions for $\mathrm{Re}\left[\Gamma(\omega_+)\right]$ and $\mathrm{Im}\left[\Gamma(\omega_+)\right]$ were worked out as an expansion in δ (keeping up to quadratic terms in δ). Substituting them along with Eq.(4.13) into Eq.(4.17) one obtains the final formula for f_n

$$f_n = \frac{AcR\omega_o}{\pi}\left\{\left(\frac{\pi\omega_r}{N\omega_o}\right)\delta^2\,\frac{M\left(\frac{\pi\omega_r}{N\omega_o}\right) - \frac{1}{2}N\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2} \right. +$$

$$\left. - \frac{\sin\left(\frac{\pi\omega_r}{N\omega_o}\right)\sin\left(\frac{\pi\omega_r}{N\omega_o}M\right)\cos\left(\frac{\pi\omega_r}{N\omega_o}(2n - M - 1)\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2}\right\} \quad . \tag{4.18}$$

Denoting the expression in curly bracket by $\tilde{f}_n$, one can introduce a dimensionless beam loading force. Its asymptotic behavior for the resonant frequencies, ω_r, in the vicinity of the integer multiples of the r.f. frequency, $N\omega_o$, namely:

$$\omega_r = k\,N\omega_o + \Delta\omega_r \ , \tag{4.19}$$

is determined by the relative strength of the expressions appearing in the denominator of Eq.(4.18). Defining the immediate vicinity of the integer multiple by the following inequality

$$\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) << \left(\frac{\pi\omega_r}{N\omega_o}\right)^2 \delta^2, \qquad (4.20)$$

one can rewrite it using Eq.(4.19) as

$$\frac{\Delta\omega_r}{N\omega_o} << k\delta \quad . \qquad (4.21)$$

If the above condition is satisfied, Eq.(4.18), reduces to the following simple expression:

$$\tilde{f}_n = M \quad , \qquad (4.22)$$

where a bunch independent beam loading force scales as the total number of bunches, M. Outside the immediate vicinity of the integer multiple (inequality, Eq.(4.21), reversed) our expression, Eq.(4.18), assumes the following asymptotic form:

$$\tilde{f}_n = - \frac{\sin\left(\frac{\pi\omega_r}{N\omega_o} M\right) \cos\left(\frac{\pi\omega_r}{N\omega_o}(2n - M - 1)\right)}{\sin\left(\frac{\pi\omega_r}{N\omega_o}\right)} \quad , \qquad (4.23)$$

which does not depend explicitly on the resonance width, δ. Apart from the integer multiples of the r.f. frequency, $N\omega_o$, the structure of Eq.(4.18) (zeros of $\sin\left(\frac{\pi\omega_r}{N\omega_o} M\right)$) reveals another finer level of symmetry governed by the fractional, $\frac{\ell}{M}$, multiples of $N\omega_o$. Indeed, the second (bunch index dependent) term in Eq.(4.18) vanishes for a discrete set of resonant frequencies defined by

$$\omega_r = \left(k + \frac{\ell}{M}\right) N\omega_o \ , \quad \ell = 1, 2, ..., M - 1 \qquad (4.24)$$

The beam loading force for these resonant frequencies does not depend on the bunch index and it scales according to the following asymptotics

$$\tilde{f}_n = O\left(\delta^2 \right) , \tag{4.25}$$

Conversely, one can find frequency regions where bunch-to-bunch variation of the beam loading force is the strongest. From Eq.(4.23) one can easily identify them with the extremes of $\sin\left(\frac{\pi\omega_r}{N\omega_o} M\right)$, namely

$$\omega_r = \left(k + \frac{2\ell + 1}{2M}\right) N\omega_o \ , \quad \ell = 1, 2,...,M - 1 \tag{4.26}$$

Figure 1 illustrates a family of curves $\tilde{f}_n(\omega_r)$ for N = 1000, M = 10 and Q =100. All the asymptotic features of the beam loading force, as discussed above, are visible in our example. To compare it with the beam loading force in case of a completely filled ring given by $\left(\tilde{f}\right)_{full}$, extracted from Eq.(4.13) as follows:

$$\left(\tilde{f}\right)_{full} = \left(\frac{\pi\omega_r}{N\omega_o}\right) N\, \delta^2 \, \frac{\left(\frac{\pi\omega_r}{N\omega_o}\right) - \frac{1}{2}\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2 \delta^2} \ , \tag{4.27}$$

we plot the above quantity for our numerical example (see Figure 2). Again, the peak value scales with the total number of bunches, N, and the width of the peak in Figure 2 is determined by the following identity:

$$\Delta\omega_r = N\omega_o \frac{k}{Q} \ . \tag{4.28}$$

5. INCOHERENT SYNCHROTRON TUNE SHIFT - POTENTIAL WELL DISTORTION

We wish to evaluate general form of the incoherent synchrotron tune shift, $\Delta\omega_n^2$, given by Eq.(3.6), where $n = 0,\ldots, M-1$ for the case of partially filled ring ($M < N$). Summing explicitly over m, one can rewrite Eq.(3.6) into the following convenient form

$$\Delta\omega_n^2 = \omega_0 \frac{A}{2\pi i} \sum_{p=-\infty}^{\infty} e^{2\pi i (p\omega_0)\frac{n}{N\omega_0}} (p\omega_0)\, Z_{\parallel}(p\omega_0)\, \frac{1 - e^{-2\pi i (p\omega_0)\frac{M}{N\omega_0}}}{1 - e^{-2\pi i (p\omega_0)\frac{1}{N\omega_0}}} = \sum_{p=-\infty}^{\infty} G(p\omega_0) \ . \tag{5.1}$$

The last part of Eq.(5.1) highlights generic sampling structure of the above formula.

The above expression, Eq.(5.1), resembles Eq.(4.1) – the starting point of the previous section; in fact one can apply exactly the same method, as used in Sec. 4, to evaluate the incoherent tune shift. Repeating a whole sequence of steps from the previous section, described by Eqs.(4.2) - (4.11), one obtains an analog of Eq.(4.12), which can be written as follows

$$\Delta\omega_n^2 = \omega_0 \frac{A}{2\pi i} N \sum_{p=-\infty}^{\infty} (Np\omega_0)\, Z_{\parallel}(Np\omega_0) + \tag{5.2}$$

$$- Ai \frac{R}{2Q}\left[\frac{\omega_+^2\, e^{2\pi i n \frac{\omega_+}{N\omega_0}}\left(1 - e^{2\pi i (N-M)\frac{\omega_+}{N\omega_0}}\right)}{\left(1 - e^{2\pi i \frac{\omega_+}{N\omega_0}}\right)\left(1 - e^{2\pi i \frac{\omega_+}{\omega_0}}\right)} - \frac{\omega_-^2\, e^{2\pi i n \frac{\omega_-}{N\omega_0}}\left(1 - e^{2\pi i (N-M)\frac{\omega_-}{N\omega_0}}\right)}{\left(1 - e^{2\pi i \frac{\omega_-}{N\omega_0}}\right)\left(1 - e^{2\pi i \frac{\omega_-}{\omega_0}}\right)}\right].$$

Similarly, the first term in Eq.(5.2) can be immediately identified with the incoherent tune shift for a fully populated ring, $(\Delta\omega^2)_{full}$, given before by Eq.(3.13). Furthermore, it was evaluated explicitly for our model resonant impedance, Eq.(4.7), in the Appendix C (see Eq.(C.12)). The result can be summarized as follows

$$(\Delta\omega^2)_{\text{full}} = \frac{1}{2}\, AR\omega_r^{\,2}\,\delta \left\{ \frac{\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right) + 2\,\delta^2\left(\frac{\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2} - \frac{2\,\cos\left(\frac{1}{2}\tau\omega_r\right)}{\frac{1}{2}\tau\omega_r} \right\} \quad . \tag{5.3}$$

Here, τ is a characteristic bunch length in units of time. Using previously defined function $\Gamma(\omega)$, Eq.(4.14), one can express $\Delta\omega_n^2$ in the following compact form

$$\Delta\omega_n^2 = (\Delta\omega^2)_{\text{full}} - Ai\,\frac{R}{2Q}\left[\omega_+^{\,2}\,\Gamma(\omega_+) - \omega_-^{\,2}\,\Gamma(\omega_-)\right] \quad . \tag{5.4}$$

Employing symmetry of $\Gamma(\omega)$, Eq.(4.16), and the fact that $\omega_- = -\omega_+^*$, one can rewrite Eq.(5.4) as follows

$$\Delta\omega_n^2 = (\Delta\omega_n^2)_{\text{full}} + A\,\frac{R}{2Q}\,2\text{Im}\left[\omega_+^{\,2}\,\Gamma(\omega_+)\right] \quad . \tag{5.5}$$

Substituting the explicit expressions for $\text{Re}\left[\Gamma(\omega_+)\right]$ and $\text{Im}\left[\Gamma(\omega_+)\right]$ along with Eq.(5.3) into Eq.(5.5) one obtains the final formula for $\Delta\omega_n^2$

$$\Delta\omega_n^2 = \frac{A\omega_o\omega_r}{\pi}\,\frac{R}{Q}\left\{ -\frac{T_o}{2\tau}\cos\left(\pi N\,\frac{\tau}{T_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)\frac{M\,\delta^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \frac{1}{4}N\,\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2} \right. +$$

$$\left. - \frac{1}{2\delta}\,\frac{\sin\left(\frac{\pi\omega_r}{N\omega_o}\right)\sin\left(\frac{\pi\omega_r}{N\omega_o}M\right)\sin\left(\frac{\pi\omega_r}{N\omega_o}(2n - M - 1)\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2} \right\} \tag{5.6}$$

Denoting the expression in curly bracket by $\Delta\tilde{\omega}_n^2$, one can introduce a dimensionless square of the synchrotron tune shift. Its asymptotic behavior for the resonant frequencies, ω_r, in the vicinity of the integer multiples of the r.f. frequency, $N\omega_o$, namely:

$$\omega_r = k\,N\omega_o + \Delta\omega_r \ , \tag{5.7}$$

is determined by the relative strength of the expressions appearing in the denominator of Eq.(5.6). Defining the immediate vicinity of the integer multiple by the following inequality

$$\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) << \left(\frac{\pi\omega_r}{N\omega_o}\right)^2 \delta^2, \tag{5.8}$$

one can rewrite it using Eq.(5.7) as

$$\frac{\Delta\omega_r}{N\omega_o} << k\delta \quad . \tag{5.9}$$

If the above condition is satisfied, Eq.(5.6), reduces to the following simple expression:

$$\Delta\tilde{\omega}_n{}^2 = -\frac{T_o}{2\tau}\cos\left(\pi N \frac{\tau}{T_o}\right) + M \quad , \tag{5.10}$$

where a bunch independent square of the synchrotron frequency shift scales as the total number of bunches, M. Assuming a realistic value of a bunch length, in terms of the bunching factor, $N\frac{\tau}{T_o}$, equal to 0.1, the first term in Eq.(5.10) can be evaluated approximately as follows

$$\frac{T_o}{2\tau}\cos\left(\pi N \frac{\tau}{T_o}\right) \approx 5N \quad . \tag{5.11}$$

Outside the immediate vicinity of the integer multiple (inequality, Eq.(5.9), reversed) our expression, Eq.(5.6), assumes the following asymptotic form:

$$\Delta\tilde{\omega}_n{}^2 = -5N + \frac{1}{2}N\left(\frac{\pi\omega_r}{N\omega_o}\right)\cot\left(\frac{\pi\omega_r}{N\omega_o}\right) - \frac{Q\sin\left(\frac{\pi\omega_r}{N\omega_o}M\right)\sin\left(\frac{\pi\omega_r}{N\omega_o}(2n - M - 1)\right)}{\sin\left(\frac{\pi\omega_r}{N\omega_o}\right)} \quad , \tag{5.12}$$

which does not depend explicitly on the resonance width, δ. Apart from the integer multiples of the r.f. frequency, $N\omega_o$, the structure of Eq.(5.6) (zeros of $\sin\left(\frac{\pi\omega_r}{N\omega_o}M\right)$) reveals another finer level of symmetry governed by the fractional, $\frac{\ell}{M}$, multiples of $N\omega_o$. Indeed, the third (bunch index dependent) term in Eq.(5.6) vanishes for a discrete set of resonant frequencies defined by

$$\omega_r = \left(k + \frac{\ell}{M}\right) N\omega_o \ , \quad \ell = 1, 2, \ldots, M - 1 \tag{5.13}$$

The amount of the synchrotron tune shift for these resonant frequencies does not depend on the bunch index. Conversely, one can find frequency regions where bunch-to-bunch variation of the synchrotron tune is the strongest. From Eq.(5.12) one can easily identify them with the extremes of $\sin\left(\frac{\pi\omega_r}{N\omega_o} M\right)$, namely

$$\omega_r = \left(k + \frac{2\ell + 1}{2M}\right) N\omega_o \ , \quad \ell = 1, 2, \ldots, M - 1 \tag{5.14}$$

One can see from Eqs.(5.6) and (5.12) that the synchrotron frequency shift in case of a partially and completely filled ring are almost the same, since the third term in both equations is very small compared to the first two. Figure 3 illustrates the case of a completely filled ring (for N = 1000 and Q =100), where the synchrotron frequency shift denoted by $\left(\Delta\tilde{\omega}^2\right)_{full}$, extracted from Eq.(5.6) is plotted as a function of the resonant frequency, ω_r.

To illustrate a small bunch-to-bunch variation effect, a family of curves $\Delta\tilde{\omega}_n^2 - \left(\Delta\tilde{\omega}^2\right)_{full}$ is plotted as a function of ω_r in Figure 4 (N = 1000, M = 10 and Q =100). All the asymptotic features of the synchrotron tune shift, as discussed above, are visible in our example.

6. SUMMARY

Both quantities, the beam loading force and the incoherent synchrotron tune shift, were calculated analytically (using contour integration technique) for a general situation of a partially filled ring ($M < N$), where individual bunches are mutually interacting via wake fields generated by resonant structures. Resulting simple analytic formulas describe the beam loading force and the incoherent tune shift experienced by a given bunch within the train, as a function of the resonant frequency, ω_r, and the quality factor of the coupling impedance, Q.

The first formula reveals resonant frequency regions in the vicinity of the integer multiples of the r.f. frequency, $N\omega_o$, where the beam loading response is still equal for all bunches (its absolute value scales as M). It also identifies the second set of characteristic resonant frequencies, spaced by the multiples of $N\omega_o/M$, at which the beam loading force is not only bunch independent, but also considerably smaller (it scales as MQ^{-2}). Conversely, our analytic formula identifies frequency regions, where bunch-to-bunch variation of the beam loading force is the strongest (ω_r at odd multiples of $N\omega_o/2M$). Similarly, an analogous analytic formula describing the amount of incoherent synchrotron tune shift suffered by different bunches was derived.

Both analytic expressions give one an insight into various optimizing schemes; e.g. to modify the existing configuration of parasitic cavity resonances (via frequency tuning), so that the resulting bunch-to-bunch spread of the beam loading force and/or the synchrotron tune spread could be instrumental in stabilizing some unstable modes of the coupled bunch instability. A number of other possible applications of the presented formalism emerge from the fact that for a given configuration of cavity resonances one can get immediately a simple quantitative answer in terms of the beam loading and the synchrotron tune shift experienced by each bunch along the train.

Superimposing many parasitic cavity modes one can use the above formulas to choose appropriate tuning of existing configuration of parasitic modes to increase stability of the coupled bunch motion.

APPENDIX A

The following useful summation identity can be proven by inspection

$$\sum_{p=-\infty}^{\infty} F(p\omega_0) = \sum_{q=-\infty}^{\infty} \sum_{\ell=0}^{N-1} F\Big((Nq+\ell)\omega_0\Big)\,, \tag{A.1}$$

where F is an arbitrary continuous function and N is a positive integer. Indeed, any integer p can be written as $p = Nq + \ell$, where the numbers q and ℓ are unique (there is one-to-one correspondence between p and a pair (q,ℓ)). Therefore summation over p is equivalent to a double summation over q and ℓ.

Let us evaluate the following sum of the first M N-th roots of unity, $M \le N$

$$A_{\mu\nu} = \sum_{m=0}^{M-1} e^{2\pi i m \frac{\mu-\nu}{N}}\,, \qquad \begin{array}{l} \mu = 0\ldots N-1 \\ \nu = 0\ldots N-1 \end{array} \tag{A.2}$$

The above sum is in fact a sum of a geometric series, which can be easily evaluated as

$$A_{\mu\nu} = e^{\pi i \frac{\mu-\nu}{N}(M-1)} \frac{\sin \pi(\mu-\nu)\frac{M}{N}}{\sin \pi \frac{\mu-\nu}{N}}\,. \tag{A.3}$$

One can see that for $M = N$, the above equation simplifies as follows

$$A_{\mu\nu} = \sum_{m=0}^{N-1} e^{2\pi i m \frac{\mu-\nu}{N}} = N\,\delta_{\mu\nu}\,. \tag{A.4}$$

We notice in passing that the general form of $A_{\mu\nu}$, given by Eq.(A.3), depends on the difference of μ and ν. Therefore, one can identify expression B_ℓ, (see Section 2), with $A_{\ell 0}$.

$$B_\ell = A_{\ell 0}\,, \tag{A.5}$$

APPENDIX B

We wish to calculate the following pair of integrals:

$$I^{+}(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, Z_{\parallel}(\omega)\, \frac{e^{2\pi i k \frac{\omega}{N\omega_o}}}{1 - e^{-2\pi i \frac{\omega}{N\omega_o}}} \;, \qquad k \geq 0 \tag{B.1}$$

and

$$I^{-}(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, Z_{\parallel}(\omega)\, \frac{e^{-2\pi i k \frac{\omega}{N\omega_o}}}{1 - e^{-2\pi i \frac{\omega}{N\omega_o}}} \;, \qquad k \geq 0 \tag{B.2}$$

Assuming general form of the longitudinal impedance of a resonant structure, given Eq.(4.7), one can rewrite it in the following form

$$Z_{\parallel}(\omega) = -iR\, \frac{\omega_r}{Q}\, \frac{\omega}{\left(\omega - \omega_+\right)\left(\omega - \omega_-\right)} \;, \tag{B.3}$$

where the singularities of $Z_{\parallel}(\omega)$ are defined by the following pair of complex poles $\omega_{\pm}$ (located in the upper half plane)

$$\omega_{\pm} = \omega_r(\pm 1 + i\delta), \qquad \delta = \frac{1}{2Q} << 1 \quad . \tag{B.4)}$$

Here R is the shunt impedance, Q is the quality factor of the resonator and ω_r is its resonant frequency.

Both integrants, written explicitly in Eqs.(B1) and (B2), have the same configuration of singularities in the complex ω-plane. It includes a pair of poles, $\omega_{\pm}$, in the upper half-plane, introduced by $Z_{\parallel}(\omega)$ and an infinite array of poles located on the real axis at integer multiples of $N\omega_o$. This last set of poles is introduced by the zeros of the following denominator: $1 - e^{-2\pi i \frac{\omega}{N\omega_0}}$, which appears in both Eqs.(B1) and (B2). Furthermore, the integrant of $I^+(k)$ restricted to a semi-circle of radius R, closed in the upper half-plane, vanishes exponentially with $R \to \infty$ (faster than $\frac{1}{R}$). Similar property holds for the integrant of $I^-(k)$ in the lower half plane.

Now we are ready to employ Cauchy's integral theorem to evaluate principle value integrals $I^+(k)$ and $I^-(k)$ explicitly. Figure 5 illustrates a complete set of singularities along with the appropriate choice of integration contours for both $I^+(k)$ and $I^-(k)$; C^+ and C^- respectively. Carrying out integration along these contours via Cauchy's integral theorem reduces integrals $I^+(k)$ and $I^-(k)$ to the following sum of residuum

$$I^+(k) = \frac{1}{2}\,\omega_0\,\frac{1}{2\pi i}\,N\sum_{p=-\infty}^{\infty} Z_\parallel(Np\omega_0) - i\,\frac{R}{2Q}\left[\frac{\omega_+\,e^{2\pi i k\frac{\omega_+}{N\omega_0}}}{1 - e^{-2\pi i\frac{\omega_+}{N\omega_0}}} - \frac{\omega_-\,e^{2\pi i k\frac{\omega_-}{N\omega_0}}}{1 - e^{-2\pi i\frac{\omega_-}{N\omega_0}}}\right] , \tag{B.5}$$

and

$$I^-(k) = -\frac{1}{2}\,\omega_0\,\frac{1}{2\pi i}\,N\sum_{p=-\infty}^{\infty} Z_\parallel(Np\omega_0) \ , \tag{B.6}$$

The infinite sum over p appearing in both Eqs.(B.5) and (B.6) can be rewritten via the Poisson sum identity, Eq.(4.2) as follows

$$N\omega_0\,\frac{1}{2\pi i}\sum_{p=-\infty}^{\infty} Z_\parallel(Np\omega_0) = \sum_{q=-\infty}^{\infty}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, e^{2\pi i q\frac{\omega}{N\omega_0}}\, Z_\parallel(\omega) \ , \tag{B.7}$$

Since both singularities of $Z_\parallel$, $\omega_\pm$, are located in the upper half plane the following integral

$$I_q = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, e^{2\pi i q\frac{\omega}{N\omega_0}}\, Z_\parallel(\omega) \ , \tag{B.8}$$

vanishes identically for q < 0. Simple application of Cauchy's integral theorem along the contour C, illustrated in Figure 4, yields the following expression for I_q, if q > 0

$$I_q = -\,i\,\frac{R}{2Q}\left[\,\omega_+\, e^{2\pi i k \frac{\omega_+}{N\omega_o}} - \omega_-\, e^{2\pi i k \frac{\omega_-}{N\omega_o}}\right] \quad , \text{for } q > 0. \tag{B.9}$$

The remaining nontrivial case of I_q (q = 0) can be evaluated via Cauchy's integral theorem as follows

$$I_0 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, \mathrm{Re}\Big[Z_{\|}(\omega)\Big] = -\,i\omega_r\,\frac{R}{2Q}\,. \tag{B.10}$$

Substituting Eqs.(B.9) and (B.10) into Eq.(B.7) one can carry out the remaining summation over q employing the following convergence formula

$$\sum_{q=1}^{\infty} e^{2\pi i q \frac{\omega_\pm}{N\omega_o}} = \frac{e^{2\pi i \frac{\omega_\pm}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_\pm}{N\omega_o}}} \quad , \quad \text{since } \mathrm{Im}\,\omega_\pm > 0. \tag{B.11}$$

The resulting expression can be summarized as follows

$$N\omega_0\,\frac{1}{2\pi i}\sum_{p=-\infty}^{\infty} Z_{\|}(Np\omega_0) = -\,i\,\frac{R}{2Q}\left[\omega_r + \omega_+\,\frac{e^{2\pi i \frac{\omega_+}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_+}{N\omega_o}}} - \omega_-\,\frac{e^{2\pi i \frac{\omega_-}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_-}{N\omega_o}}}\right] \quad , \tag{B.12}$$

where

$$\omega_\pm = \omega_r(\pm 1 + i\delta), \qquad \delta = \frac{1}{2Q}\;. \tag{B.13}$$

After some algebra one can rewrite Eq.(B.12) into the following convenient form

$$N\omega_0\,\frac{1}{2\pi i}\sum_{q=-\infty}^{\infty} Z_{\|}(Nq\omega_0) = -\,i\omega_r\,\delta^2\,\frac{\left(\frac{\pi\omega_r}{N\omega_o}\right) - \sin\left(\frac{\pi\omega_r}{N\omega_o}\right)\cos\left(\frac{\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2\delta^2}\;. \tag{B.14}$$

APPENDIX C

We wish to evaluate an analog of Eq.(B.7), given by the following expression

$$J = N\omega_0 \frac{1}{2\pi i} \sum_{p=-\infty}^{\infty} (Np\omega_0)\, Z_{\|}(Np\omega_0)\ , \tag{C.1}$$

The infinite sum over p appearing in Eq.(C.1) can be rewritten via the Poisson sum identity, Eq.(4.2) as follows

$$J = \sum_{q=-\infty}^{\infty} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, e^{2\pi i q \frac{\omega}{N\omega_0}}\, \omega\, Z_{\|}(\omega)\ , \tag{C.2}$$

Since both singularities of $Z_{\|}$, $\omega_{\pm}$, are located in the upper half plane the following integral

$$J_q = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, e^{2\pi i q \frac{\omega}{N\omega_0}}\, \omega\, Z_{\|}(\omega)\ , \tag{C.3}$$

vanishes identically for $q < 0$ (see contour C_- illustrated in Figure 6b). Simple application of Cauchy's integral theorem along the contour C_+, illustrated in Figure 6a, yields the following expression for J_q, if $q > 0$

$$J_q = -\, i \frac{R}{2Q} \left[\, \omega_+^{\,2}\, e^{2\pi i k \frac{\omega_+}{N\omega_0}} - \omega_-^{\,2}\, e^{2\pi i k \frac{\omega_-}{N\omega_0}} \right]\ , \text{ for } q > 0. \tag{C.4}$$

The remaining nontrivial case of J_q ($q = 0$) can be written as follows

$$J_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\ \ \omega\ \mathrm{Im}\big[Z_{\|}(\omega)\big]. \tag{C.5}$$

One can notice that the above integral diverges for our model impedance, given by Eq.(B.3). Indeed, for large values of ω, $\mathrm{Im}\left[Z_{\parallel}(\omega)\right] \sim \frac{1}{\omega}$, therefore, $J_0 \sim \int_{-\infty}^{\infty} d\omega\ \omega\ \frac{1}{\omega} \rightarrow \infty$. This unphysical divergence can be removed assuming finite bunch length rather than a point like bunch structure. Assuming a simple rectangular bunch of length τ (in time units) one should redefine J_0 as follows

$$\tilde{J}_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\ \ \rho(\omega)\ \ \omega\ \mathrm{Im}\left[Z_{\parallel}(\omega)\right]. \tag{C.6}$$

where the bunch spectrum is given by

$$\rho(\omega) = \frac{\sin\left(\frac{1}{2}\ \omega\tau\right)}{\frac{1}{2}\ \omega\tau}\ . \tag{C.7}$$

Simple application of Cauchy's integral theorem along the contours $C_{\pm}$, illustrated in Figure 6, yields the following expression for $\tilde{J}_0$

$$\tilde{J}_0 = -\ i\omega_r\ \frac{R}{2Q}\ \frac{\cos\left(\frac{1}{2}\ \omega_r\tau\right)}{\frac{1}{2}\ \omega_r\tau}\ . \tag{C.8}$$

Substituting Eqs.(C.4) and (C.8) into Eq.(C.2) one can carry out the remaining summation over q employing the following convergence formula

$$\sum_{q=1}^{\infty} e^{2\pi i q \frac{\omega_{\pm}}{N\omega_o}} = \frac{e^{2\pi i \frac{\omega_{\pm}}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_{\pm}}{N\omega_o}}}\ , \quad \text{since } \mathrm{Im}\ \omega_{\pm} > 0. \tag{C.9}$$

The resulting expression can be summarized as follows

$$J = i\frac{R}{2Q}\left[i\omega_r^{\ 2}\ \frac{\cos\left(\frac{1}{2}\ \omega_r\tau\right)}{\frac{1}{2}\ \omega_r\tau} - \omega_+^{\ 2}\ \frac{e^{2\pi i \frac{\omega_+}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_+}{N\omega_o}}} + \omega_-^{\ 2}\ \frac{e^{2\pi i \frac{\omega_-}{N\omega_o}}}{1 - e^{2\pi i \frac{\omega_-}{N\omega_o}}} \right]\ , \tag{C.10}$$

where

$$\omega_{\pm} = \omega_r(\pm 1 + i\delta), \qquad \delta = \frac{1}{2Q} \ . \tag{C.11}$$

After some algebra one can rewrite Eq.(C.10) into the following convenient form

$$J = \frac{1}{2} R\omega_r^{\,2}\,\delta \left\{ \frac{\sin\left(\frac{2\pi\omega_r}{N\omega_o}\right) + 2\,\delta^2 \left(\frac{\pi\omega_r}{N\omega_o}\right)}{\sin^2\left(\frac{\pi\omega_r}{N\omega_o}\right) + \left(\frac{\pi\omega_r}{N\omega_o}\right)^2 \delta^2} - \frac{2\,\cos\left(\frac{1}{2}\tau\omega_r\right)}{\frac{1}{2}\tau\omega_r} \right\} \ . \tag{C.12}$$

REFERENCES

1. A.W. Chao, Physics of Collective Beam Instabilities in High Energy Accelerators, (John Wiley & Sons, Inc. 1993) Ch. IV,

2. K.A. Thompson and R.D. Ruth, Proceedings of the 1989 IEEE Particle Accelerator Conference, 792

ACKNOWLEDGMENT

The author wishes to thank Bill Ng and Pat Colestock for stimulating discussions and comments.

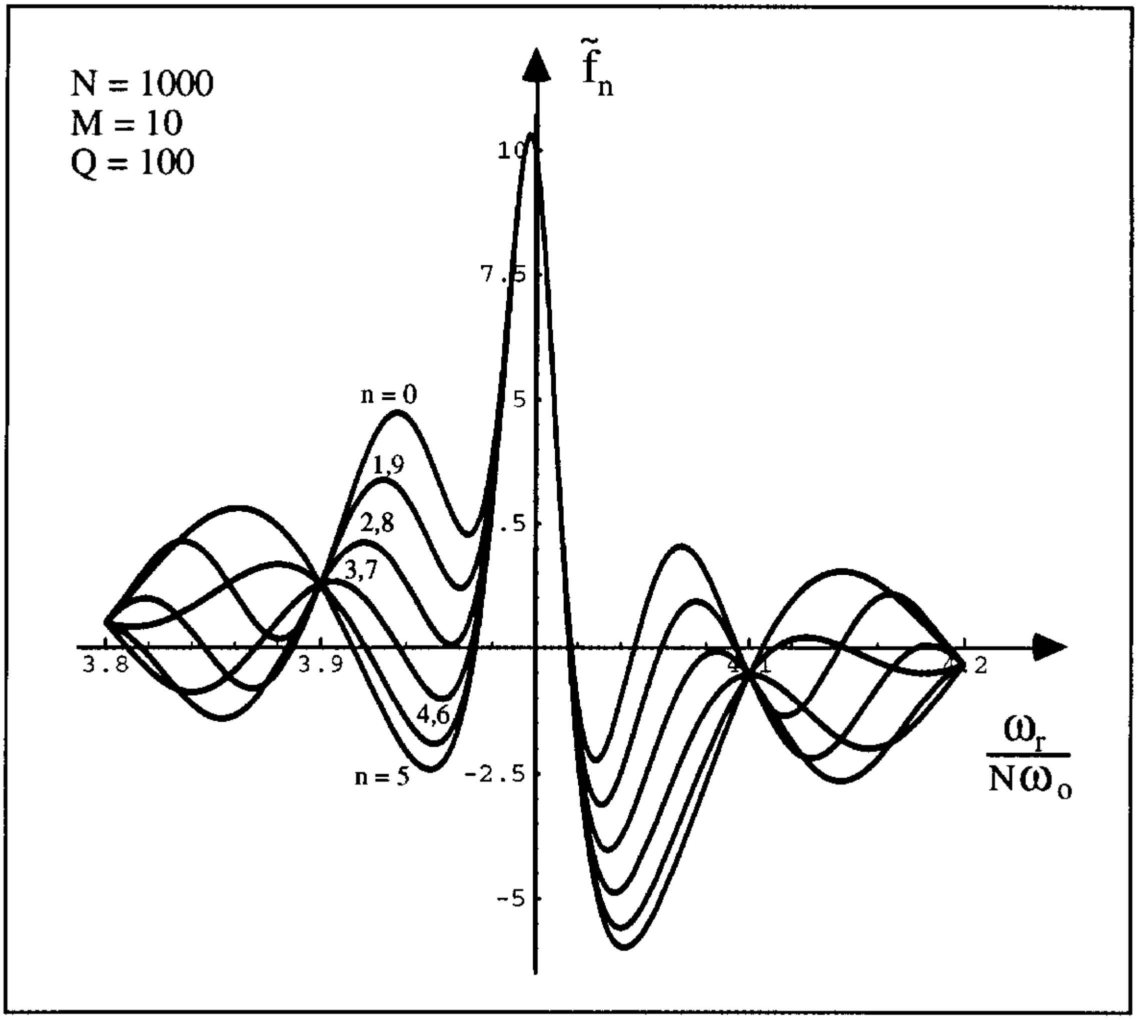

N = 1000
M = 10
Q = 100
$\tilde{f}_n$
10
7.5
5
2.5
-2.5
-5
n = 0
1,9
2,8
3,7
4,6
n = 5
3.8
3.9
4.1
4.2
$\frac{\omega_r}{N\omega_o}$

Figure 1

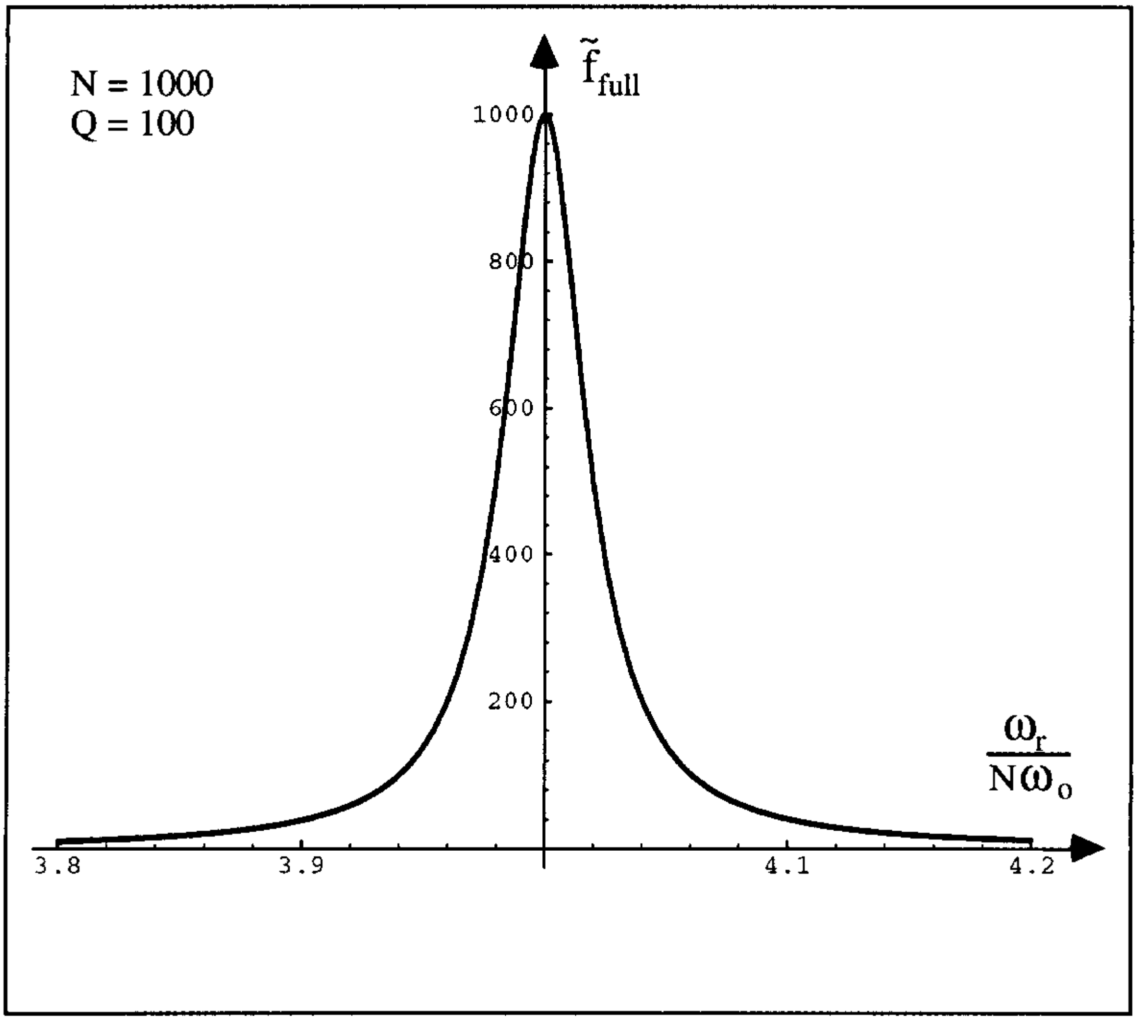
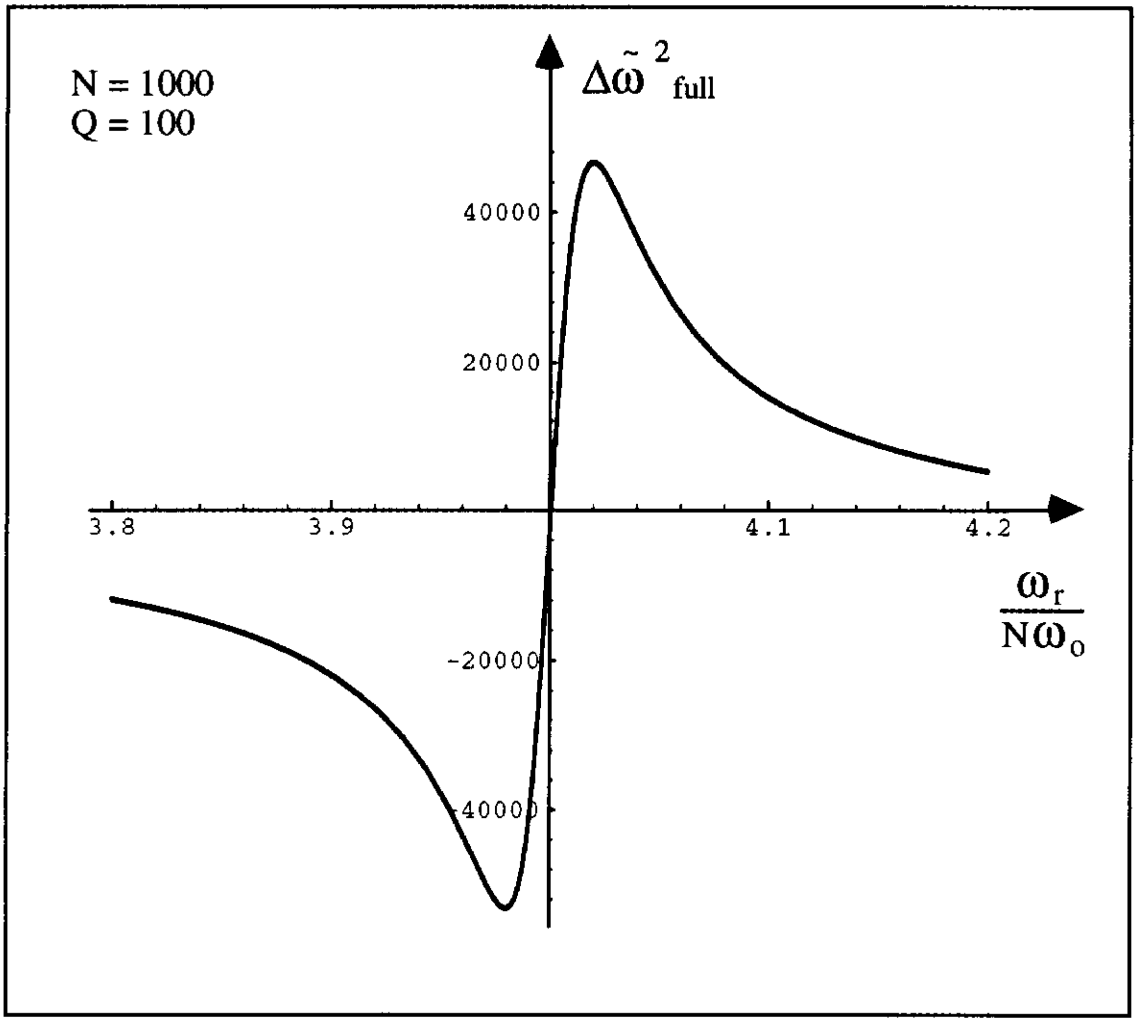
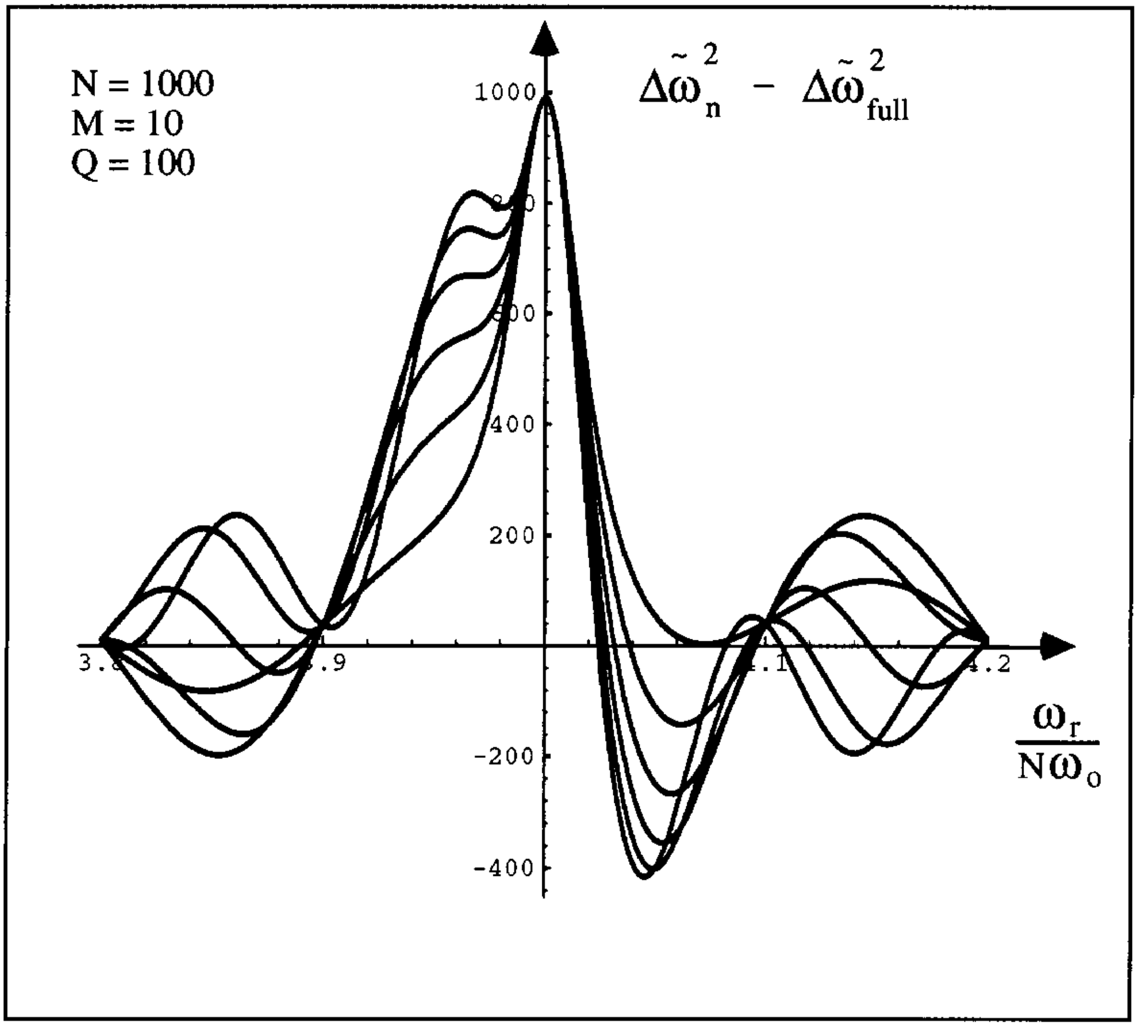

N = 1000
Q = 100
$\tilde{f}_{full}$
1000
800
600
400
200
$\frac{\omega_r}{N\omega_o}$
3.8
3.9
4.1
4.2

Figure 2



N = 1000
Q = 100
$\Delta\tilde{\omega}^2_{full}$
40000
20000
-20000
-40000
3.8
3.9
4.1
4.2
$\frac{\omega_r}{N\omega_o}$

Figure 3



N = 1000
M = 10
Q = 100
$\Delta\tilde{\omega}_n^2 - \Delta\tilde{\omega}_{full}^2$
1000
400
200
-200
-400
.9
.2
$\frac{\omega_r}{N\omega_o}$

Figure 4

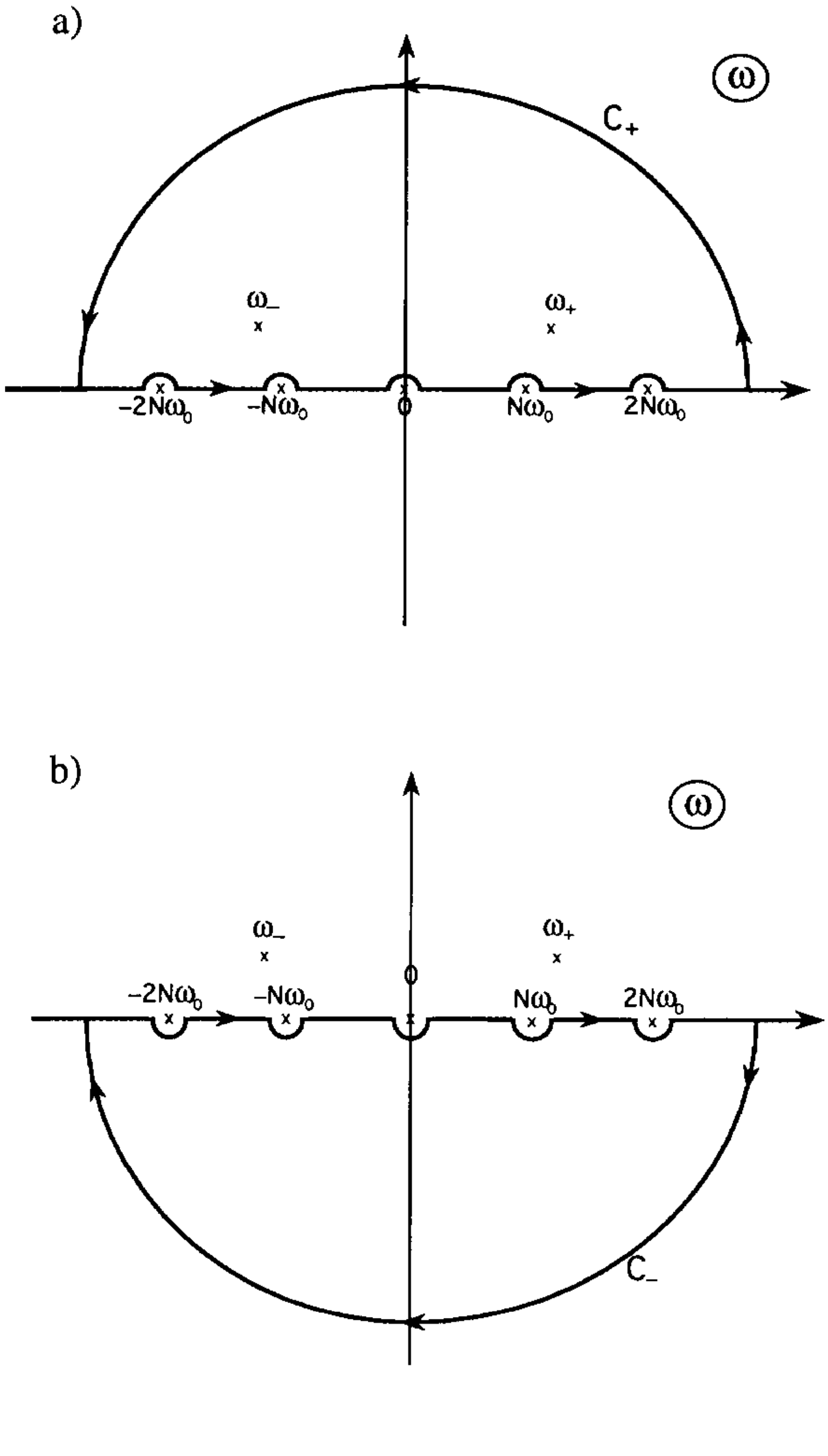


Figure 5

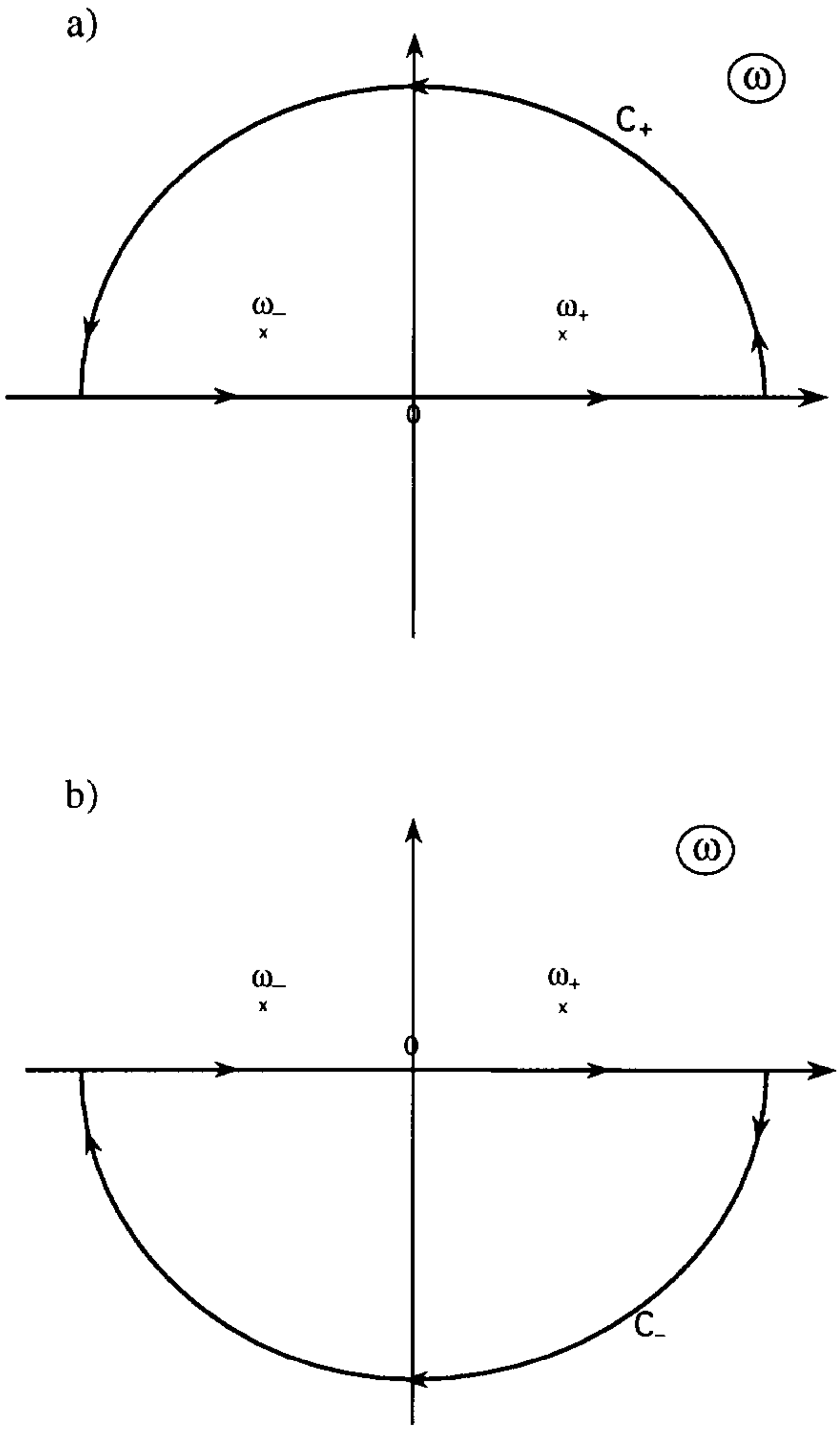


Figure 6